MATH2101 Complex Analysis (Year 2011/12)
Examination questions and solutions

The following notation is used throughout: for any $a \in \mathbb{C}$ and $r > 0$, $S(a,r)$ denotes a positively oriented circular contour of radius r, centred at a; $D(a,r) = \{z : |z-a| < r\}$, $\overline{D}(a,r) = \{z : |z-a| \le r\}$, $D'(a,r) = \{z : 0 < |z-a| < r\}$.

1 Section A

1. Find the Laurent expansion of $f(z) = \dfrac{1}{z^3 - z}$ in the following domains:

   (a) $0 < |z| < 1$,

   (b) $1 < |z-1| < 2$.

   Solution.

   (a) We have
   $$\frac{1}{z^3 - z} = \frac{1}{z(z^2-1)} = -\frac{1}{z(1-z^2)}.$$
   Expand using the geometric series:
   $$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n},$$
   which converges for $|z| < 1$. Thus, the Laurent expansion of f for $0 < |z| < 1$ is
   $$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty} z^{2n} = -\sum_{n=0}^{\infty} z^{2n-1}.$$

   (b) Rewrite the function $f(z)$:
   $$\frac{1}{z^3 - z} = \frac{1}{z-1}\frac{1}{z(z+1)} = \frac{1}{z-1}\left(\frac{1}{z} - \frac{1}{z+1}\right).$$

We have

$$\frac{1}{z+1} = \frac{1}{2+z-1} = \frac{1}{2\left(1+\frac{z-1}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n,$$

which converges for $|z-1|<2$. Also,

$$\frac{1}{z} = \frac{1}{1+z-1} = \frac{1}{(z-1)\left(1+\frac{1}{z-1}\right)} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-1}\right)^n,$$

which converges for $|z-1|>1$. Thus, the Laurent expansion of f for $1 < |z-1| < 2$ is

$$f(z) = \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-1}\right)^n - \frac{1}{2(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+2}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^{n-1}.$$

2. Let a function f be analytic on the punctured disk $D'(a,r)$ with some $a \in \mathbb{C}$ and $r>0$. Describe three types of isolated singularities of the function f by explaining how they are related to the principal part of its Laurent expansion at the point a.

Solution. If a function f has an isolated singularity at a point z_0, then in a punctured neighbourhood of z_0 it has the Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n.$$

If $c_k = 0$ for $k < -M, M > 0$ and $c_{-M} \neq 0$, then the function is said to have a pole of order M.

If there is no such number $N \in \mathbb{Z}$ that $c_k = 0$ for all $k < N$, then the singularity is said to be essential.

If $c_k = 0$ for all $k < 0$, then the singularity is said to be removable.

3. (a) Define the residue of the function having an isolated singularity $a \in \mathbb{C}$.
   (b) Evaluate the residues of the function $f(z) = \frac{1}{\sin z}$ at the points $z = 0, \pi/2, \pi$.

Solution.

(a) The residue of the function f is defined as the coefficients c_{-1} in the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k.$$

(b) Let us find the Principal Part (PP) of $f(z)$ at each of the points. Since

$$\sin z = z - \frac{z^3}{6} + \dots,$$

at $a = 0$ we have

$$f(z) = \frac{1}{z - \frac{z^3}{6} + \dots} = \frac{1}{z\left(1 - \frac{z^2}{6} + \dots\right)} = \frac{1}{z}\left(1 + \frac{z^2}{6} + \dots\right).$$

The Principal Part equals $\frac{1}{z}$, so the residue equals 1.
The function f is holomorphic at $\pi/2$, and hence PP $= 0$, so $Res(f, \frac{\pi}{2}) = 0$.
Consider $a = \pi$. Expand $\sin z$ in the powers of $(z - \pi)$:

$$\sin z = -\sin(z-\pi) = -(z-\pi) + \frac{(z-\pi)^3}{6} + \dots.$$

Therefore, as a few lines before,

$$f(z) = -\frac{1}{(z-\pi) - \frac{(z-\pi)^3}{6} + \dots} = -\frac{1}{z-\pi}\left(1 + \frac{(z-\pi)^2}{6} + \dots\right).$$

Thus the Principal Part equals $-\frac{1}{z-\pi}$, and hence the residue is -1.

4. Using an appropriate substitution and the Cauchy Integral Formula, evaluate the integral

$$\int_0^{2\pi} e^{e^{it}}\, dt.$$

Solution. Substitute $z = e^{it}$, so that $dt = -iz^{-1}dz$. Thus

$$\int_0^{2\pi} e^{e^{it}}\,dt = -i \int_{S(0,1)} e^z \frac{dz}{z}.$$

By the Cauchy formula the integral equals

$$2\pi i \cdot (-i)e^0 = 2\pi.$$

5. Can one find an entire function g satisfying the property:

$$g\left(\frac{1}{n}\right) = g\left(-\frac{1}{n}\right) = \frac{1}{n^2},$$

for all $n = 1, 2, \ldots$? If yes, how many such functions are there? Justify your answer.

Solution.

The function g is given by $g(z) = z^2$, by inspection. By the Unique Continuation Theorem there is only one such function.

6. Determine all complex values z for which the series

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n} \right)$$

converges.

Solution. Split the series in two:

$$A(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad B(z) = \sum_{n=0}^{\infty} \frac{n^2}{z^n}.$$

The first series gives the exponential function $\exp z$, and it is known to converge for all $z \in \mathbb{C}$.

To study the second series define $w = \frac{1}{z}$, so that

$$B(z) = \sum_{n=0}^{\infty} n^2 w^n.$$

The radius of convergence of this series is 1, so $B(z)$ converges absolutely for $|w| < 1$, i.e. $|z| > 1$, and diverges for $|z| < 1$. If $|z| = 1$, then $|n^2 z^n| \to \infty$ as $n \to \infty$, so the series diverges. Therefore, the total series converges absolutely for $|z| > 1$ and diverges for $|z| \leq 1$.

2 Section B

7. (a) Suppose that f is holomorphic on the punctured disk $D'(a,r)$, $a \in \mathbb{C}, r > 0$, and that $|f(z)| \leq M$ for some constant $M > 0$ and all $z \in D'(a,r)$. Prove that a is a removable singularity of f.

   (b) Let g be an entire function such that $|g(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that g is constant.

Solution.

(a) Compute the coefficients c_{-n} with $n \geq 1$ by taking $\rho \in (0,r)$ and integrating:

$$c_{-n} = \frac{1}{2\pi i} \int\limits_{S(a,\rho)} f(z)(z-a)^{n-1} dz.$$

Let us now estimate the modulus using the known estimation result:

$$|c_{-n}| \leq \frac{1}{2\pi} M \rho^{n-1} 2\pi\rho = M\rho^n.$$

Since $\rho \in (0,r)$ is arbitrary and $\rho^n \to 0$ as $\rho \to 0$, we have $c_{-n} = 0$. Thus the principal part of f is zero, and hence the singularity is removable.

(b) Define $h(z) = \frac{1}{g(z)}$. Since g has no roots, the function h is entire as well. Moreover, $|h(z)| \leq 1$, $z \in \mathbb{C}$. By Liouville's Theorem $h(z) = const$, and hence $g(z) = const$, as claimed.

8. (a) Let g be an entire function. Show that one cannot have the $|g^{(n)}(0)| \geq n^n n!$ for *all* $n = 1, 2, \ldots$.

   (b) Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)^2} dx,$$

   where $a > 0$.

Solution.

(a) Suppose that the inequality holds for all n. Thus the Taylor series

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \tag{1}$$

has the zero radius of convergence. Indeed,

$$\left| \frac{g^{(n)}(0)}{n!} \right| \geq n^n, \ n \geq 1.$$

The series $\sum_n n^n |z|^n$ has the zero radius of convergence, and by the comparison principle, so does (1). Therefore g cannot be entire.

(b) Let

$$f(z) = \frac{e^{iz}}{(z^2 + a^2)^2},$$

so that the sought integral equals $\operatorname{Re} I$, where

$$I = \int_{\mathbb{R}} f(x) dx.$$

The function f has double poles at $z_\pm = \pm ia$, but only z_+ in the upper half-plane. Let

$$\Gamma_R^{(1)} = \{z : \operatorname{Im} z = 0, -R \leq \operatorname{Re} z \leq R\},$$
$$\Gamma_R^{(2)} = \{z : z = Re^{it}, \ t \in [0, \pi]\},$$
$$\Gamma_R = \Gamma_R^{(1)} \cup \Gamma_R^{(2)}.$$

By the Cauchy Residue Theorem,

$$\begin{aligned}\int_{\Gamma_R} f(z)dz &= 2\pi i \text{ Res }(f, z_+) = 2\pi i \frac{d}{dz}(z-z_+)^2 f(z)\Big|_{z=z_+} \\ &= 2\pi i \frac{d}{dz}\frac{e^{iz}}{(z+ia)^2}\Big|_{z=ia} = 2\pi i \left(\frac{ie^{iz}}{(z+ia)^2} - \frac{2e^{iz}}{(z+ia)^3}\right)\Big|_{z=ia} \\ &= 2\pi i e^{-a}\left(-\frac{i}{4a^2} - \frac{i}{4a^3}\right) = \pi e^{-a}\frac{a+1}{2a^3}.\end{aligned}$$

Furthermore, by the Jordan's Lemma,

$$\int_{\Gamma_R^{(2)}} f(z)dz \to 0, \ R \to \infty.$$

Therefore

$$\int_{\mathbb{R}} f(z) = \lim_{R\to\infty} \int_{\Gamma_R^{(1)}} f(z)dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = \pi e^{-a}\frac{a+1}{2a^3},$$

and hence

$$I = \text{Re}\left(\pi e^{-a}\frac{a+1}{2a^3}\right) = \pi e^{-a}\frac{a+1}{2a^3}.$$

9. (a) Let f be an even entire function, i.e. $f(z) = f(-z)$ for all $z \in \mathbb{C}$.
   the Taylor expansion of f at $a = 0$ contains only even powers of z.

   (b) Using an appropriate substitution and the Cauchy Residue Theorem, show that

$$\int_0^{2\pi} \frac{\cos\theta}{13 + 12\cos\theta}\, d\theta = -\frac{4\pi}{15}.$$

Solution.

(a) Write the Taylor expansion of the function f:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

so that

$$f(-z) = \sum_{k=0}^{\infty} c_k (-1)^k z^k,$$

Since the function is even, we have

$$2f(z) = f(z) + f(-z) = 2 \sum_{\text{even } k} c_k z^k,$$

as claimed.

(b) Denote the integral by I. Make the following substitution: $z = e^{i\theta}$, so that

$$d\theta = -i\frac{dz}{z}, \ \cos\theta = \frac{z + z^{-1}}{2}.$$

Thus

$$\begin{aligned} I = & -i \int_{S(0,1)} \frac{z^2+1}{z(26 + 12z + 12z^{-1})} \frac{dz}{z} \\ = & -i \int_{S(0,1)} \frac{z^2+1}{z(26z + 12z^2 + 12)} dz. \end{aligned}$$

Factorise the polynomial in the denominator:

$$26z + 12z^2 + 12 = 12\left(z + \frac{3}{2}\right)\left(z + \frac{2}{3}\right).$$

Therefore

$$I = -\frac{i}{12}\int_{S(0,1)} \frac{z^2+1}{\left(z+\frac{3}{2}\right)\left(z+\frac{2}{3}\right)}\frac{dz}{z}.$$

The integrand

$$f(z) = \frac{z^2+1}{z\left(z+\frac{3}{2}\right)\left(z+\frac{2}{3}\right)}$$

has three poles, two of which, 0 and $-2/3$ are inside the unit circle. Thus by the Cauchy Residue Theorem,

$$I = 2\pi i \cdot \left(-\frac{i}{12}\right)\left[Res(f,0) + Res(f,-2/3)\right].$$

The pole at 0 is simple, so that

$$Res(f,0) = \lim_{z\to 0} \frac{z^2+1}{\left(z+\frac{3}{2}\right)\left(z+\frac{2}{3}\right)} = 1.$$

The pole at $-2/3$ is also simple, so

$$Res(f,-2/3) = \lim_{z\to -2/3} \frac{z^2+1}{z\left(z+\frac{3}{2}\right)} = -\frac{2^2+3^2}{3^2\cdot\frac{2}{3}\left(-\frac{2}{3}+\frac{3}{2}\right)} = -\frac{13}{5}.$$

Therefore

$$I = -\frac{\pi}{6}\cdot\frac{8}{5} = -\frac{4\pi}{15}.$$